Partial Fractions decompositions

(continued)

Evaluate: $\displaystyle\int \frac{1}{x^2-7x+10}\,dx$ (reducible quadratic)

reducible

$$= \int \frac{1}{(x-2)(x-5)}\,dx \quad \text{(distinct linear factors)}$$

$$= \int \frac{1}{(x-2)(x-5)}\,dx = \int \frac{A}{x-2}\,dx + \int \frac{B}{x-5}\,dx$$

· Find A and B.

Since $\dfrac{1}{(x-2)(x-5)} = \dfrac{A}{x-2} + \dfrac{B}{x-5}$

Multiply each term by LCD: $(x-2)(x-5)$

$$\Longrightarrow \quad 1 = A(x-5) + B(x-2)$$

set $x=5$ (ie plug 5 into the expression)

$$1 = A(0) + B(5-2)$$

$$1 = 3B \longrightarrow \boxed{B = \tfrac{1}{3}}$$

Likewise, set $x=2 \Longrightarrow 1 = A(-3) + 0B$

$$\longrightarrow \boxed{A = -\tfrac{1}{3}}$$

Therefore

$$\int \frac{1}{x^2-7x+10}\,dx = \int \frac{A}{x-2}\,dx + \int \frac{B}{x-5}\,dx$$

$$= \int \frac{-\frac{1}{3}}{x-2}\,dx + \int \frac{\frac{1}{3}}{x-5}\,dx$$

$$= -\frac{1}{3}\int \frac{1}{x-2}\,dx + \frac{1}{3}\int \frac{1}{x-5}\,dx$$

$$= -\frac{1}{3}\ln|x-2| + \frac{1}{3}\ln|x-5| + C$$

$$\underline{\text{Or}} = \ln\left|\frac{(x-5)^{\frac{1}{3}}}{(x-2)^{\frac{1}{3}}}\right| + C \quad \underline{\text{or}} \quad \ln\left|\sqrt[3]{\frac{x-5}{x-2}}\right| + C$$

Example $\int \frac{3x-9}{(x-1)(x+2)^2}\,dx$ (repeated linear factor)

$$\frac{3x-9}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$\Longrightarrow 3x-9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$$

. Set $x=1 \longrightarrow 3(1)-9 = A(1+2)^2 + B(0) + C(0)$

$$-6 = 9A \longrightarrow \boxed{A = \frac{-2}{3}}$$

. set $x=-2 \longrightarrow 3(-2)-9 = A(0) + B(-3)(0) + C(-2-1)$

$$\longrightarrow -15 = -3C \longrightarrow \boxed{C = +5}$$

. set $x=0$ (since "+1" and "-2" values of x, eliminate "B")

$$3(0)-9 = A(2)^2 + B(-2) + C(-1)$$

$$-9 = \left(\frac{-2}{3}\right)(4) - 2B + (5)(-1)$$

$$-9 = \frac{-8}{3} - 5 - 2B \longrightarrow -9 + \frac{23}{3} = -2B$$

$$\longrightarrow \frac{-4}{3} = -2B$$

$$\longrightarrow +2/3 = B$$

Therefore $\frac{-2/3}{x-1} + \frac{2/3}{x+2} + \frac{5}{(x+2)^2}$

$$\text{So } \int \frac{3x-9}{(x-1)(x+2)^2}dx = \int \frac{-2/3}{x-1}dx + \int \frac{2/3}{x+2}dx + \int \frac{5}{(x+2)^2}dx$$

$$= -\frac{2}{3}\ln|x-1| + \frac{2}{3}\ln|x+2| - \frac{5}{x+2} + C$$

Example,

$$\int \frac{18}{(x+3)(x^2+9)}dx$$

quadratic

irreducible

$$= \int \frac{A}{x+3}dx + \int \frac{Bx+C}{x^2+9}dx$$

Now: $\dfrac{18}{(x+3)(x^2+9)} = \dfrac{A}{x+3} + \dfrac{Bx+C}{x^2+9}$

$\longrightarrow 18 = A(x^2+9) + (Bx+C)(x+3)$

set $x=-3 \longrightarrow 18 = 18A \longrightarrow \boxed{A=1}$

$18 = 1(x^2+9) + (Bx+C)(x+3)$

$18 = x^2+9 + Bx^2+3Bx + Cx + 3C$

$0x^2+0x+18 = x^2(B+1) + x(3B+C) + 9+3C$

$$\Longrightarrow \begin{cases} B+1=0 & \longrightarrow \boxed{B=-1} \\ 3B+C=0 & \longrightarrow C=-3B \longrightarrow \boxed{C=3} \\ 9+3C=18 & \end{cases}$$

$$\int \frac{18}{(x+3)(x^2+9)}dx = \int \frac{1}{x+3}dx + \int \frac{-x+3}{x^2+9}dx$$

$$= \ln|x+3| + \int \frac{-x}{x^2+9}dx + \int \frac{3}{x^2+9}dx$$

$$= \ln|x+3| - \frac{1}{2}\ln|x^2+9| + \arctan\left(\frac{x}{3}\right) + C$$

Example: what if we had

$$\int \frac{18}{(x+3)(x^2-9)}dx$$

reducible!

$$\Longrightarrow \int \frac{18}{(x+3)(x-3)(x+3)}dx = \int \frac{18}{(x+3)^2(x-3)}dx$$

$$\frac{18}{(x+3)^2(x-3)} = \frac{A}{x-3} + \frac{B}{(x+3)} + \frac{C}{(x+3)^2}$$

$$\Longrightarrow 18 = A(x+3)^2 + B(x-3)(x+3) + C(x-3)$$

- set $x=3$, $18 = 36A + 0B + 0C \rightarrow \boxed{A = \frac{1}{2}}$
- set $x=-3$, $18 = 0A + 0B - 6C \rightarrow \boxed{C = -3}$
- set $x=0$, $18 = 9A - 9B - 3C$

$$18 = \frac{9}{2} - 9B + 9$$

$$9 - \frac{9}{2} = -9B \rightarrow \frac{9}{2} = -9B \rightarrow \boxed{B = -1/2}$$

Thus, $\displaystyle\int \frac{1/2}{x-3}dx + \int \frac{-1/2}{x+3}dx + \int \frac{-3}{(x+3)^2}dx$

$$= \frac{1}{2}\ln|x-3| - \frac{1}{2}\ln|x+3| + \frac{3}{x+3} + C$$